MATHEMATICAL METHODS UNITS 3 AND 4

2.05 – Rational functions

If P(x) and Q(x) are polynomials in x, then

$$f(x) = \frac{P(x)}{Q(x)} \text{ where } Q(x) \neq 0$$

is called a **rational function**.

If the degree (i.e. – highest power of x) of P(x) < the degree of Q(x), then the rational function f(x) is said to be a **proper rational function**.

Example 1

$f(x) = \frac{1}{x}$ and $g(x) = \frac{x^2+3}{x^5+3x^2-2}$ are proper rational functions.

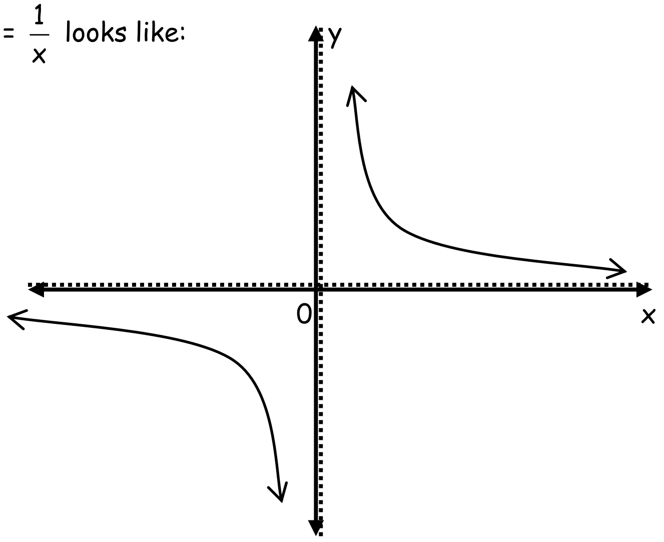
$f(x) = \frac{3x}{x+2}$ and $g(x) = \frac{x^2}{x-2}$ are improper rational functions.

Sketching Rational functions

Translations parallel to the x-axis

Consider the family of curves with equation $f(x) = \frac{1}{x+a}$ for $a \in R$

The graph of $f(x) = \frac{1}{x}$ looks like:



This graph is known as a hyperbola, and the dotted lines are asymptotes.
An asymptote is a line that the graph 'tends to', but never reaches. Asymptotes are values on a graph that you **cannot** have. Here the asymptotes are $x = 0$, $y = 0$.

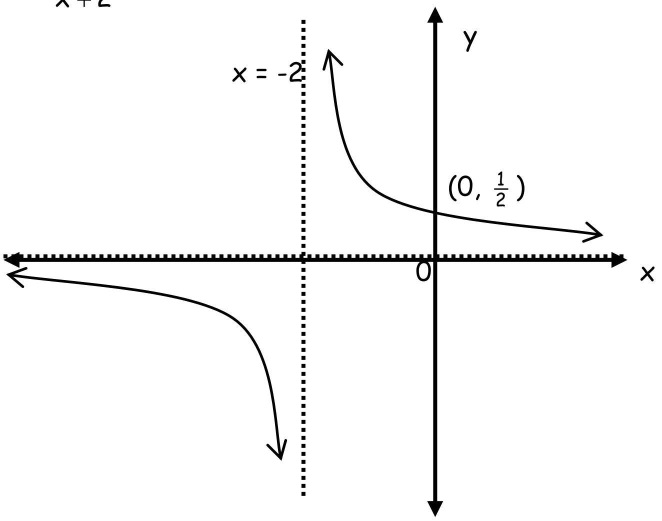
Guidelines for sketching graphs with asymptotes

- Asymptotes occur when the denominator (bottom line) = 0 or when the rational function has a constant out the front, otherwise it is at y = 0.
- Asymptotes must be shown as dotted lines, particularly when they are also an axis.
- The sketch must always 'tend towards' the asymptote, and (in general terms) shouldn't cross it

Example 2

Sketch the graph of $f(x) = \frac{1}{x+2}$, $x \neq -2$

The graph of $f(x) = \frac{1}{x+2}$, $x \neq -2$ looks like:



The asymptotes are now x = -2. and y = 0. The graph is not defined when x = -2.

Consider the graph of $f(x) = \frac{1}{x+a}$, $x \neq -a$ If $a > 0$ the graph of $y = \frac{1}{x+a}$ is obtained by translating the graph of $y = \frac{1}{x}$ by 'a' units parallel to the x-axis in the negative direction. If $a < 0$ the graph of $y = \frac{1}{x-a}$ is obtained by translating the graph of $y = \frac{1}{x}$ by 'a' units parallel to the x-axis in the positive direction.

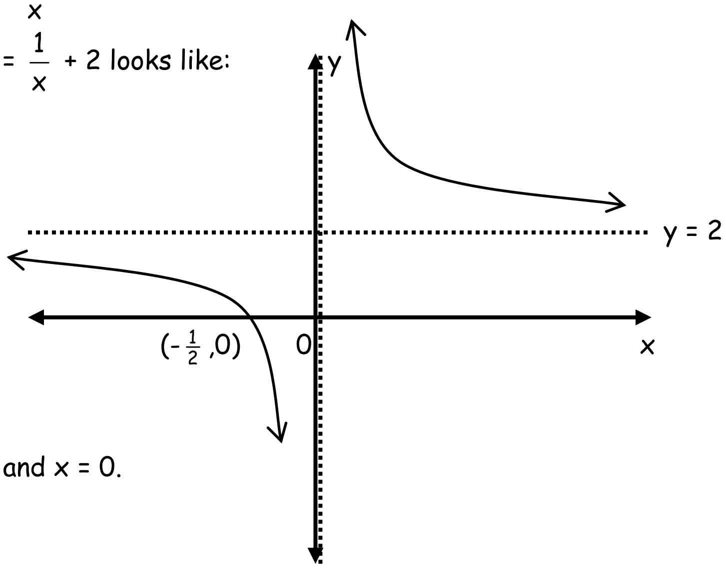
Translations parallel to the y-axis

Consider the family of curves $y = \frac{1}{x} + a$, where $a \in R$.

Example 3

Sketch the graph of $f(x) = \frac{1}{x} + 2$

The graph of $f(x) = \frac{1}{x} + 2$ looks like:



The asymptotes are y = 2 and x = 0.

In general: the graph of $f(x) = \frac{1}{x} + a$ is obtained by translating the graph of $f(x) = \frac{1}{x}$, '+a' units parallel to the y axis, and the graph of $f(x) = \frac{1}{x} - a$, is a translation of the $f(x) = \frac{1}{x}$ graph '-a' units parallel to the y- axis

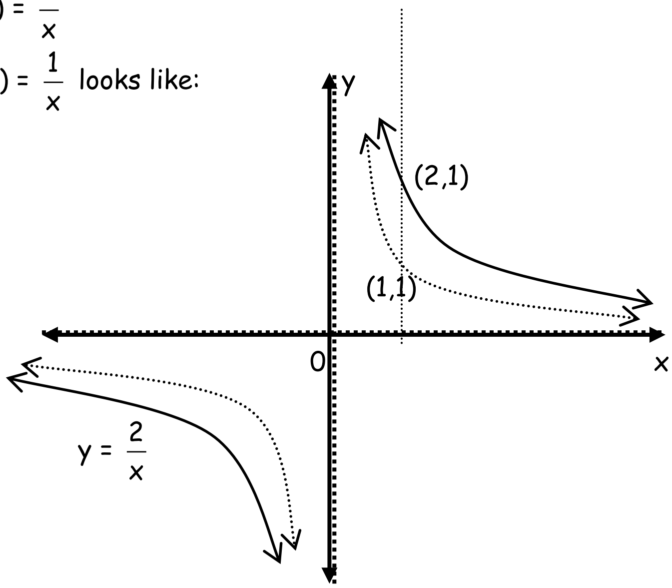
Dilations

Consider the curves in the form $y = \frac{a}{x}$.

Example 4

Sketch the graph of $f(x) = \frac{2}{x}$

The graph of $f(x) = \frac{1}{x}$ looks like:



The graph of y = af(x), a >0, is obtained by multiplying all the y-coordinates by 'a'. This is a dilation of factor 'a' from the x-axis.

If $f(x) = a + \frac{k}{x-b}$, then

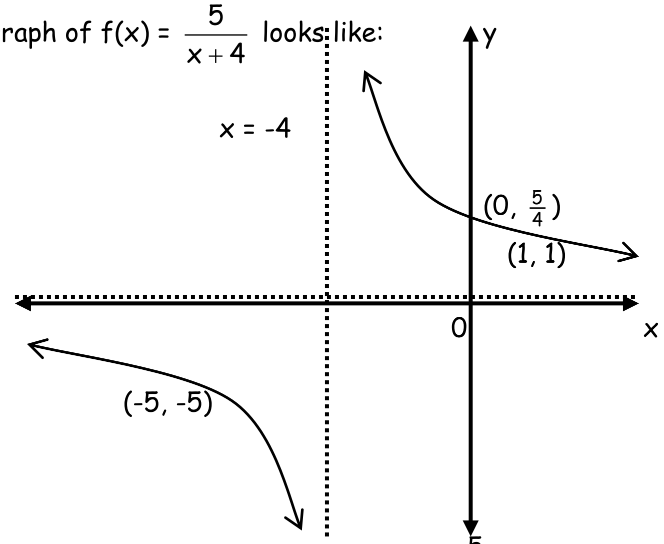
x = b is a vertical asymptote and
y = a is a horizontal asymptote.

The graph can be drawn by translating the graph of $y = \frac{k}{x}$ by 'b' units to the right and 'a' units upward.

Example 5

Sketch the graph of
$$xy + 4y - 5 = 0$$
$$xy + 4y = 5$$
$$y(x + 4) = 5$$
$$y = \frac{5}{x+4}$$ i.e. a = 0, b = -4, k = 5.

The graph of $f(x) = \frac{5}{x+4}$ looks like:



You need to show all intercepts, $\therefore$ when x = 0, $y = \frac{5}{4}$. Sometimes it is appropriate to include a few more points to show the shape of the graph. Only use 'nice' points, eg. x = -1, 1, etc.

Example 6

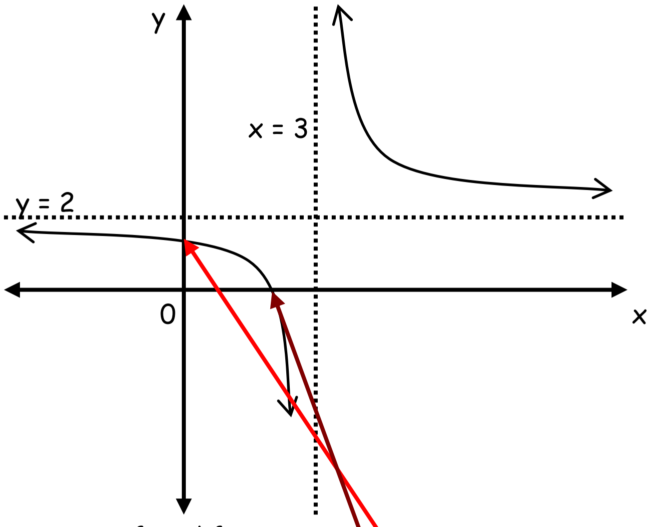
Sketch the graph of $y = \frac{1}{x-3} + 2$.

Here a = 2, b = -3 and k = 1.

This infers that the basic $\frac{1}{x}$ graph is moved up 2 units, and 3 units to the right.

The asymptotes will be x = 3, and y = 2.

The graph of $f(x) = \frac{1}{x-3} + 2$ looks like:



The axes intercepts are found from:

When x = 0, $y = -\frac{1}{3} + 2 = \frac{5}{3}$ $\therefore$ (0, $\frac{5}{3}$)

When y = 0 $\frac{1}{x-3} + 2 = 0$ $\therefore$ $\frac{1}{x-3} = -2$ $\therefore$ $x - 3 = -\frac{1}{2}$

$\therefore$ $x = 2\frac{1}{2}$ $\therefore$ $(2\frac{1}{2}, 0)$

Example 7

Sketch the graph of $y = \frac{x+4}{x-3}$ (this is an improper rational function)

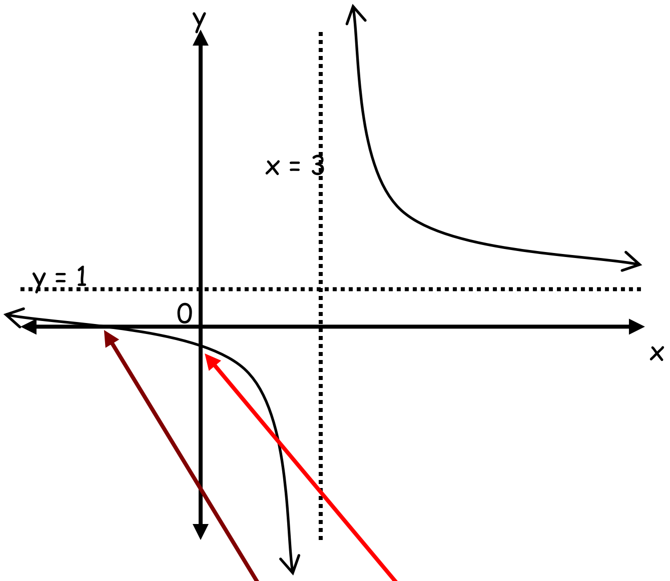
By long division

$$\begin{array}{r} 1 \\ x - 3 \;\overline{)\,x+4} \\ \underline{x - 3} \\ 7 \end{array}$$

i.e. $\frac{x+4}{x-3} = 1 + \frac{7}{x-3}$ $\therefore y = 1 + \frac{7}{x-3}$

Here a = 1, b = 3 and k = 7.

The graph of $f(x) = 1 + \frac{7}{x-3}$ looks like:



The axes intercepts are found from:

When x = 0, $y = 1 + \frac{7}{-3} = -\frac{4}{3}$ $\therefore (0, -\frac{4}{3})$

When y = 0 $1 + \frac{7}{x-3} = 0$

$\therefore \frac{7}{x-3} = -1$

$\therefore x - 3 = -7$

$\therefore x = -4$ $\therefore (-4, 0)$

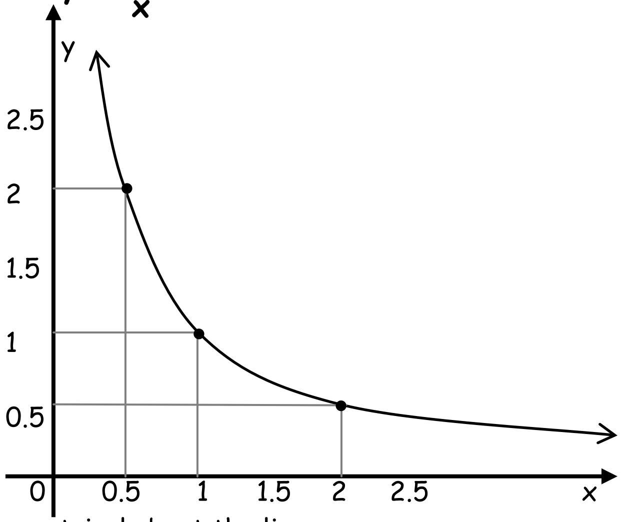
Alternatively, you can use the original expression (because it is identical)

$\therefore$ let y = 0 in $y = \frac{x+4}{x-3}$ gives us $0 = \frac{x+4}{x-3}$ $\therefore 0 = x + 4 \therefore x = -4$ $\therefore$ Intercept is (-4, 0)

Let x = 0 gives $y = \frac{0+4}{0-3} = -\frac{4}{3}$ $\therefore (0, -\frac{4}{3})$

The rectangular hyperbola

Graph of $y = \frac{1}{x}$



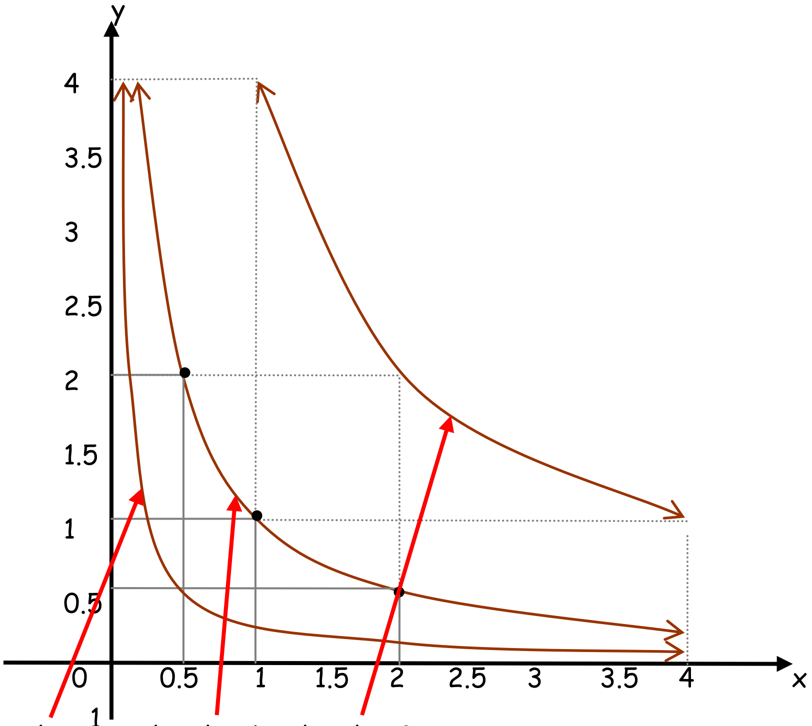
- The graph is symmetrical about the line y = x.
- The areas of the rectangles generated by each point on the curve are equal to 1 square unit. This is the reason the curve is called a rectangular hyperbola.

A rational function of the form

$$f(x) = \frac{mx + c}{m_1x + c_1}$$ is said to be a **rectangular hyperbola.**

Special case: $y = \frac{k^2}{x}$

The k is squared so that it is always positive.



$y = \frac{k^2}{x}$ when $k = \frac{1}{2}$, when k = 1, when k = 2

- In the equation $y = \frac{k^2}{x}$, the parameter 'k' defines the shape and position of the curve.
- The curve is symmetrical about the line $y = x$
- As the value of 'k' increases, the curve becomes less convex and moves away from the origin.

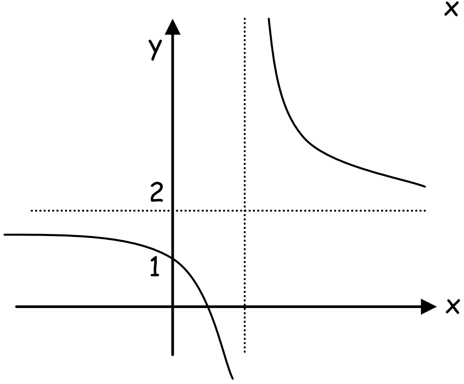
$y = \frac{k^2}{x}$ $\therefore$ $xy = k^2$. $\therefore$ the areas of the rectangles generated by each point on the curve are equal.

Example 8

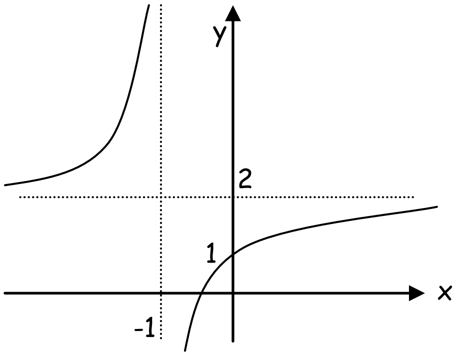
Which one of the following graphs best represents the function

$f: R\backslash\{-1\} \to R$, where $f(x) = 2 - \frac{1}{x+1}$

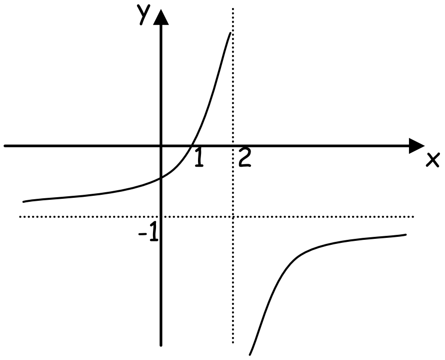
A



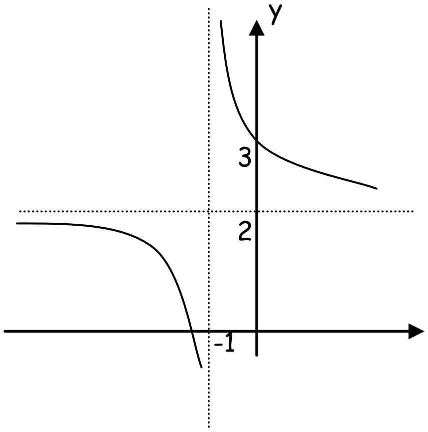
B



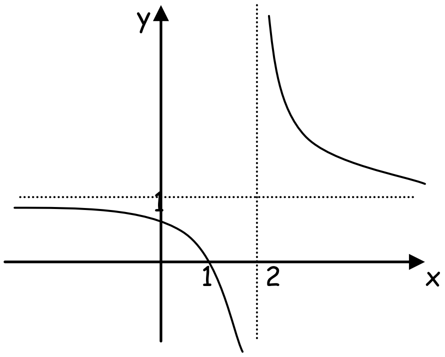
C



D



E

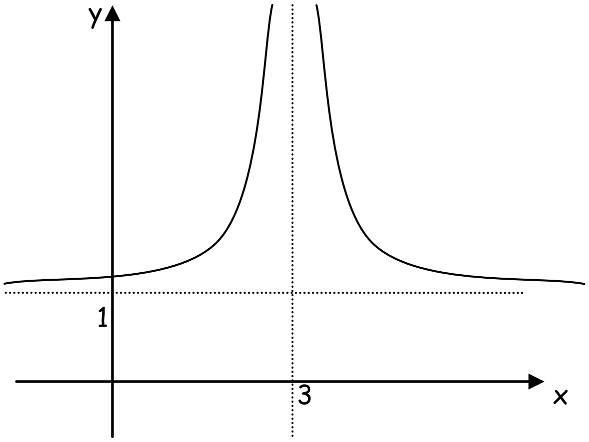


This question is best done using the graphics calculator. Plot the graph and see which looks closest.

Thinking it through; when $x = 0$, $f(x) = 2 - \frac{1}{0+1} = 1$. $\therefore$ This means that the answer is either A or B.

It is a negative hyperbola, since it has a $-\frac{1}{x+1}$ term. From the expression the horizontal asymptote needs to be y = 2. The vertical asymptote is when the denominator equals zero. This must be at x = -1. ∴ the answer is B.

Example 9



The equation of the curve shown is given by

A $y - 1 = \frac{1}{x-3}$

B $y - 1 = \frac{1}{x+3}$

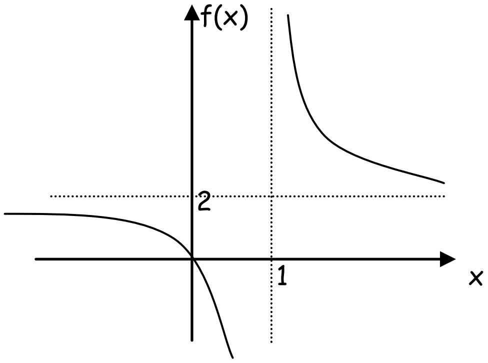
C $y - 1 = \frac{1}{(x-3)^2}$

D $y + 1 = \frac{1}{(x+3)^2}$

E $y - 3 = \frac{1}{(x-1)^2}$

From the graph, the vertical asymptote is x = 3. ∴ this infers an expression involving (x – 3).
Since the function is always positive, it must be in the form $(x - 3)^2$.
The horizontal asymptote is y = 1.
$\therefore\ y = 1 + .\frac{1}{(x-3)^2}$ this is C.

Example 10



A possible equation for the graph shown is

A $f(x) = \frac{2x-1}{x-1}$ **B** $f(x) = \frac{2x+3}{x+1}$

C $f(x) = \frac{2x}{x-1}$ **D** $f(x) = \frac{2x+4}{x+1}$ **E** $f(x) = \frac{3-2x}{x-1}$

From the graph, the vertical asymptote is at x = 1, this implies the denominator must include (x – 1).
This eliminates B and D. This is a positive graph, so E is eliminated.
The horizontal asymptote is found by turning the expression into a proper rational function, using the asymptotes give, the equation is of the form $f(x) = \frac{A}{x-1} + 2$.

Substitute the point (0, 0): $\therefore\ 0 = \frac{A}{0-1} + 2$ $\therefore\ A = 2$

$$\therefore\ f(x) = \frac{2}{x-1} + 2 = \frac{2+2(x-1)}{x-1} = \frac{2+2x-2}{x-1} = \frac{2x}{x-1} \quad \therefore\ C$$

The other option would be do sketch the 3 options on your graphics calculator.

This is the first example of an analysis task type question.

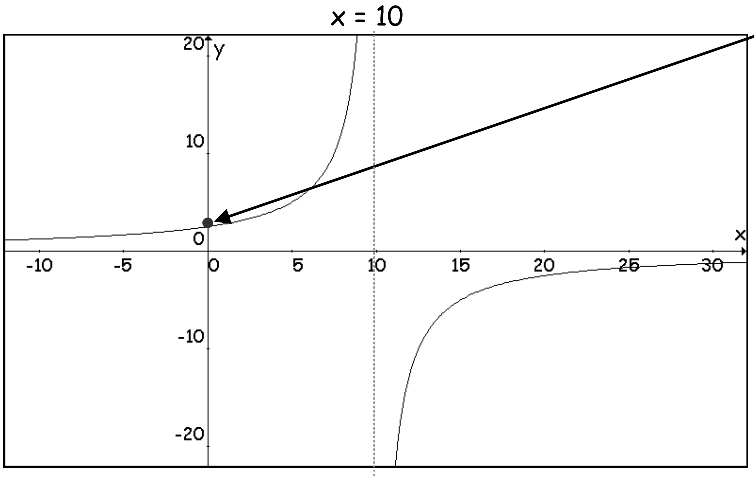
Example 11

a) On the set of axes provided below, sketch the curve with equation $f(x) = \frac{25}{10-x}$, $x \neq 10$.

Clearly show all axial intercepts and labelling all asymptotes.

This should all be done on the graphics calculator with the asymptotes being sketched on later.
The vertical axis, must be at x = 10. The horizontal asymptote is y = 0

The y-intercept, is given when x = 0. On substitution $f(x) = \frac{25}{10} = 2.5 \therefore (0, 2.5)$



The marks awarded for this question were, Shape = 1, Intercept = 1, Asymptotes = 1

b) If $g(x) = \frac{25x}{10-x}$, $x \neq 10$, show that

i. $g(x) = \frac{250}{10-x} - 25$, $x \neq 10$

Using long division we get:

$$\begin{array}{r} -25 \\ -x + 10\overline{)\,25x\quad} \\ \underline{25x - 250} \\ 250 \end{array}$$

$\therefore g(x) = -25 + \frac{250}{-x+10} = -25 + \frac{250}{10-x}$, as required

(1 mark for method, 1 mark for answer)

ii. g(x) = 10f(x) - 25

If $g(x) = -25 + \frac{250}{10-x} = -25 + 10\left(\frac{25}{10-x}\right) = -25 + 10f(x)$, as required.

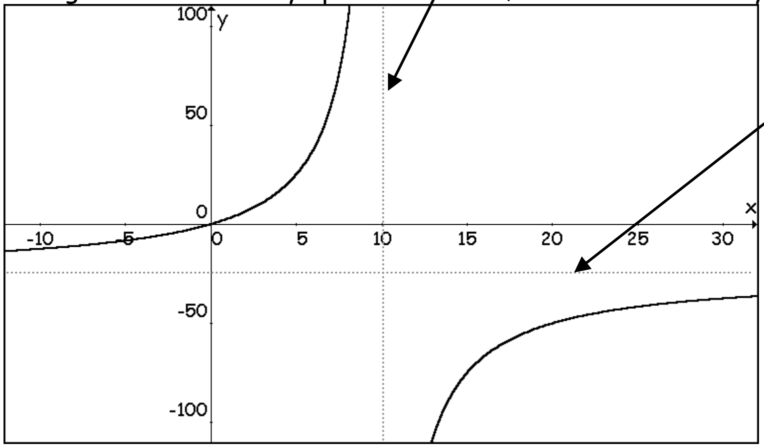
(1 mark for answer)

c) On the set of axes provided below, sketch the graph of $g(x) = \frac{25x}{10-x}$, $x \neq 10$.

This should all be done on the graphics calculator with the asymptotes being sketched on later.

You need to have the equation in the form $g(x) = \frac{250}{10-x} - 25$.

This gives a vertical asymptote at x = 10, and a horizontal asymptote at y = -25.



(1 mark for shape, 1 mark for asymptotes)

It is found that the cost, $C, in millions of dollars for a municipality to remove x% of industrial pollutants discharged into a river system is approximately modelled by

$$C(x) = \frac{25x}{100-x},\ 0 \le x \le 100$$

d) Find the cost of removing 60% of industrial pollutants discharged into a river system.

When x = 60 $C = \frac{25 \times 60}{100-60} = \frac{1500}{40} = 37.5$

∴ It would cost $37.5 million. Don't forget the millions.

(1 mark for answer)

The municipality can only allocate $10 million for the removal of industrial pollutants from their river system.

e) What percentage of pollutants remain in the river system after the clean up process?

∴ C = 10, $\rightarrow 10 = \frac{25x}{100-x}$

∴ 1000 – 10x = 25x

∴ 1000 = 35x

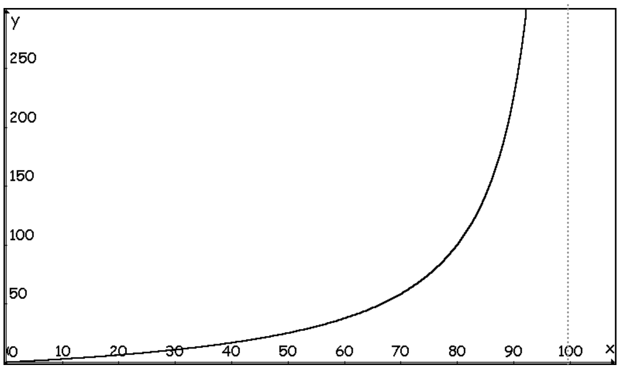
∴ x = 28.57.

i.e. 28.6% of pollutants can be removed. This means that 71.4% remain.

(1 mark for method, 1 mark for answer)

f) Sketch the graph of $C(x) = \frac{25x}{100-x}$, $0 \le x \le 100$ on the set of axes shown below.

This should all be done on the graphics calculator with the asymptotes being sketched on later. Here the vertical asymptote will be at x = 100.



(1 mark for shape, 1 mark for asymptotes)

g) Find an expression for the inverse function $C^{-1}(x)$

Let y = C(x), on interchanging x and y, we get:

$x = \frac{25y}{100 - y}$ $\therefore (100 - y)x = 25y$ $\therefore 100x - xy = 25y$

$\therefore 100x = 25y + xy$

$= y(25 + x)$

$\therefore y = \frac{100x}{25 + x}$

$\therefore C^{-1}(x) = \frac{100x}{25 + x}$ Your answer must be in this form, not y = …

(1 mark for method, 1 mark for answer)

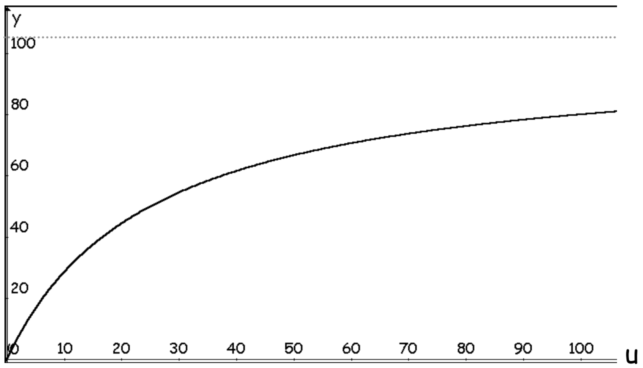
h) If N(u) = $C^{-1}(u)$, what physical property does N(u) represent?

N(u) represents the percentage of pollutants removed from the river system when $u million dollars are spent on the cleaning process.

(1 mark for answer)

i) Using part (f) sketch the graph of N(u) on the set of axes shown below?

This graph has to be reflection of graph the answer to part (f). This means that the vertical asymptote in part(f) becomes a horizontal asymptote here.



(1 mark for shape, 1 mark for asymptotes)

j) If possible, how much would it cost to completely remove all pollutants from the river system? Explain your answer.

It is impossible to remove all pollutants from the river system. (Make sure that you answer the actual question.)

(1 mark for answer)

This is because the graph of the function has an asymptote at 100%. This means that in order to get close to removing 100% of the pollutants, you would need to continually increase the funding for the cleaning process.

(1 mark for answer)

$N(u) = \frac{100u}{25 + u}$ Using long division we get: $u + 25 \overset{100}{\overline{)100u}}$

$\underline{100u + 2500}$

$- 2500$ $\therefore N(u) = 100 - \frac{2500}{u + 25}$

N(u) has a horizontal asymptote at 100, ie. 100%.