MATHEMATICAL METHODS UNITS 3 AND 4

A Review of Set Notation

A **set** is a collection of objects. The objects in the set are known as **elements.**
If x is an element of the set A, then it is written as $x \in A$. This is read as 'x is a member of the set A' or 'x belongs to A' or 'x is in A'.
The notation $x \notin A$ means x is **not** an element of A.

The elements of the set {1, 2, 3, 4, …} are called the **natural numbers**.

The elements of the set {…., -2, -1, 0, 1, 2, …} are called the **integers**.

The numbers of the form $\frac{p}{q}$ with p and q integers, $q \neq 0$, are called **rational numbers.**

The real numbers that are not which are not rationals are called **irrational** (eg π and $\sqrt{2}$)

The set of Real numbers is denoted by R.
The set of Rational numbers is denoted by Q
The set of integers is denoted by Z, (sometimes J)
The set of Natural numbers is denoted by N.

$$\therefore N \subseteq Z \subseteq Q \subseteq R$$

An **ordered pair**, denoted by (a, b), is a pair of elements a and b in which a is considered to be the first element and b the second.

Set notation is $\{x: x \in R \setminus -1\}$

The set
of x values
such that
x
is an element of
the set of Real numbers
except
-1

2.01 – Functions and their graphs

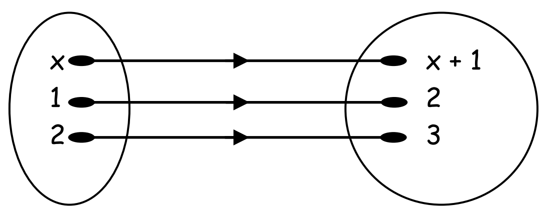
A **relation** is a set of ordered pairs.
A relation may be defined by a rule that pairs the elements. Eg. $y = 3x$

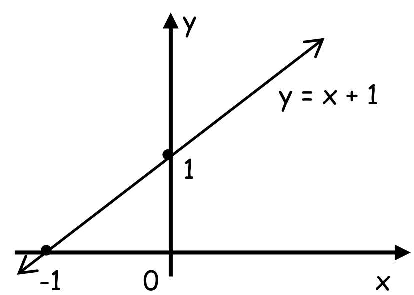
A relation is a **function** if each element of the domain determines exactly one element in the range.

A relation is a **function** if no two ordered pairs have the same first element.

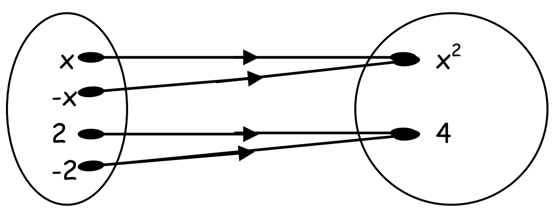
Types of relations

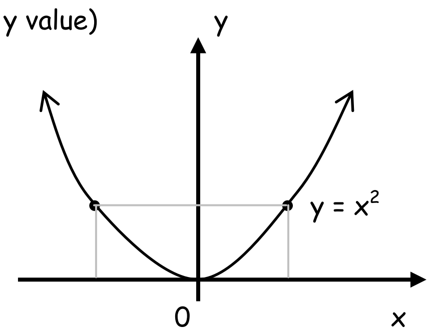
- **One to one correspondence**



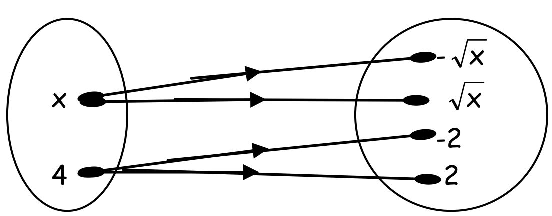


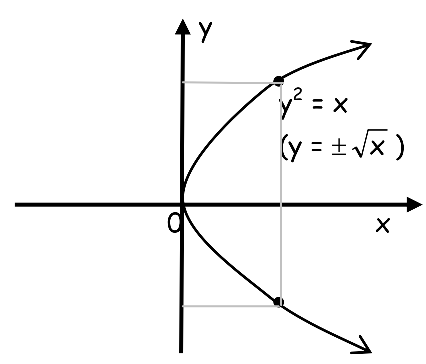
- **Many-to-one correspondence** (many x values for one y value)



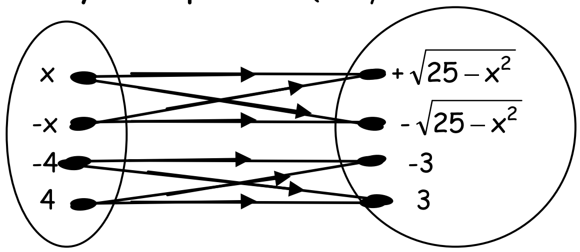


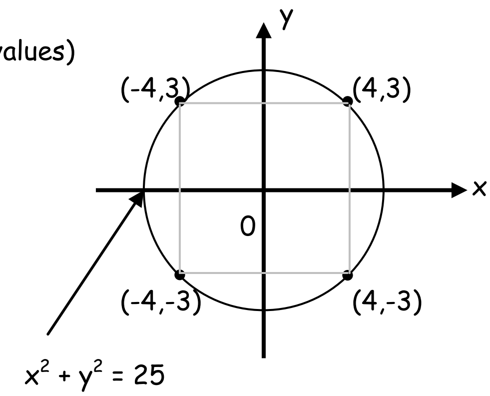
- **One-to-many correspondence** (i.e. not a function)





Many-to-many correspondence (many x values for many y values)





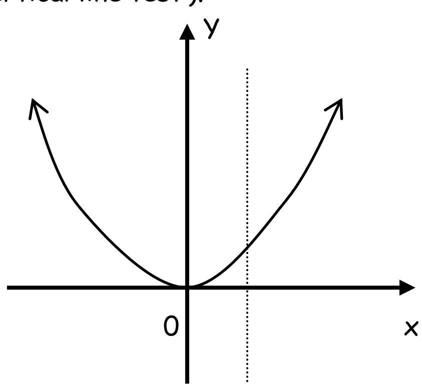
Sketching graphs

When sketching a graph, you must **always:**

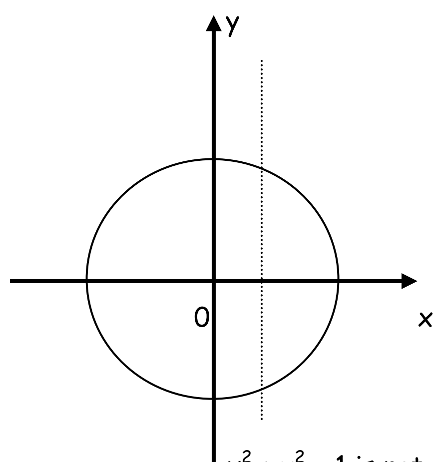
- label the axes
- Show the origin
- Include arrowheads on the axes
- Indicate +ve directions.

Functions

A **function** is a relation such that no two ordered pairs of the relation have the same first (or **x**) value. Not all relations are functions. A vertical line will cut a function at one point only. (This is known as the 'vertical line test').



$y = x^2$ is a function



$x^2 + y^2 = 1$ is not a function.

Functions will be denoted by lower case letters such as f, g, h.

The definition of functions implies that for each **x** in the domain of **f** there is a unique element, **y**. i.e. An 'x' value will give only one (unique) value of y. The element **y** is called the **image** of **x under f** or the value of **f** at **x** and is denoted by **f(x)**. **x** is called the **pre-image** of **y**.

Use your graphing calculators to solve the following, use TBLSET and TABLE. See your notes on graphing calculators for an explanation of this.

Example 1

If $f(x) = 2x^2 + x$, find f(3), f(-2) and $f(x\text{-}1)$.

$$f(3) = 2(3)^2 + (3) = 21$$
$$f(-2) = 2(-2)^2 + (-2) = 6$$
$$\begin{aligned} f(x - 1) &= 2(x - 1)^2 + (x - 1) \\ &= 2(x^2 - 2x + 1) + x - 1 \\ &= 2x^2 - 3x + 1 \end{aligned}$$

Example 2

If $f(x) = 2x + 1$, find $f(\frac{1}{a})$, $a \neq 0$.

$$f(\frac{1}{a}) = 2(\frac{1}{a}) + 1 = \frac{2}{a} + 1$$

Example 3

Consider the function defined by $f(x) = 2x - 4$ for all $x \in R$. Find the value of f(t)

$$f(t) = 2t - 4.$$

For what values of t is f(t) = t?

$$f(t) = t$$
$$2t - 4 = t$$
$$t - 4 = 0$$
$$\therefore t = 4$$

For what values of x is $f(x) \geq x$?

$$f(x) \geq x$$
$$2x - 4 \geq x$$
$$x - 4 \geq 0$$
$$\therefore x \geq 4$$

Intervals

Interval notation, a square bracket [ ] is inclusive, whereas a round bracket ( ) is exclusive.

Example 4

$(2, 4) = 2 < x < 4$

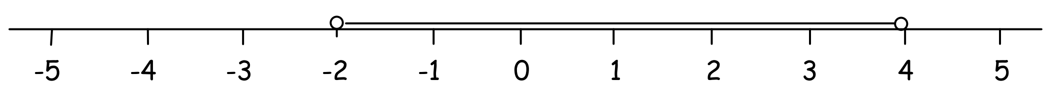
$(3, 4] = 3 < x \leq 4$

$[-2, 3] = -2 \leq x \leq 3$

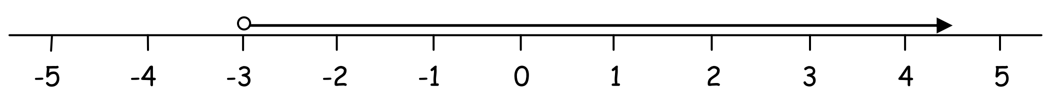
Intervals can also be represented by diagrams. The 'closed' (filled) circle indicates that the number is included. The 'open' (not filled) circle indicates that the number is not included.

Example 5

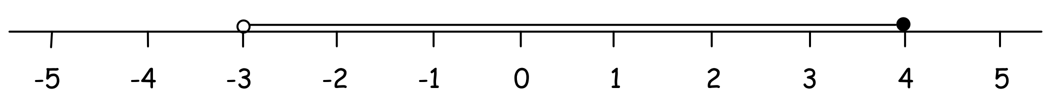
(-2, 4)



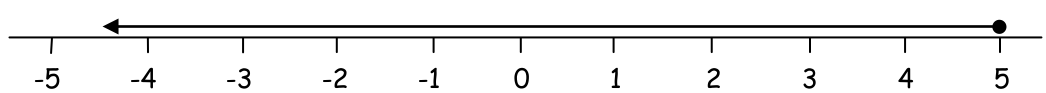
$(-3, \infty)$



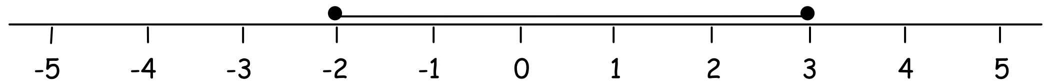
(-3, 4]



$(-\infty, 5]$



[-2, 3]



Maximal domains (implied domains)

Domain

The **domain** usually consists of all the possible **x** values (i.e. the values on the horizontal axis)
To find the domain, we express y in terms of x, and then determine the values of x for which y is defined.

Range

The **range** usually consists of all the possible "y" values (i.e. the values on the vertical axis)
To find the range, we express x in terms of y, and then determine the values of y for which x is defined.

A **restricted function** is one that has some (implied) limit on the domain.

E.g. $y = \frac{1}{(x-1)}$ does not exist when x = 1,

because $\frac{\text{anything}}{0}$ = undefined, and when $x = 1$, $\sqrt{1-1} = 0$.

$\therefore$ x > 1, gives a positive denominator.

$\{x: x \in R \setminus 1\}$

The set of x values such that x is a member of real numbers except 1.

You also need to remember that $\frac{0}{\text{anything (except 0)}} = 0$

If the domain is unspecified then the domain is the largest subset of R for which the rule is defined. For the function, $f(x) = \sqrt{x}$ the implied domain (maximal domain) is $[0, \infty)$.

We write $f: [0, \infty) \rightarrow R$, $f(x) = \sqrt{x}$

Note: ∞ and $-\infty$ always have a curved bracket next to them.

Example 6

Find the implied domain of the functions with the following rules:

$f(x) = \frac{2}{2x-3}$

$\frac{2}{0}$ is undefined

f(x) is not defined when 2x – 3 = 0, i.e. when $x = \frac{3}{2}$.

Thus the implied domain is $\{x: x \in R \setminus \frac{3}{2}\}$

$g(x) = \sqrt{5-x}$

can't find the $\sqrt{\ }$ of a negative number

g(x) is defined when $5 - x \geq 0$, i.e. when $x \leq 5$.

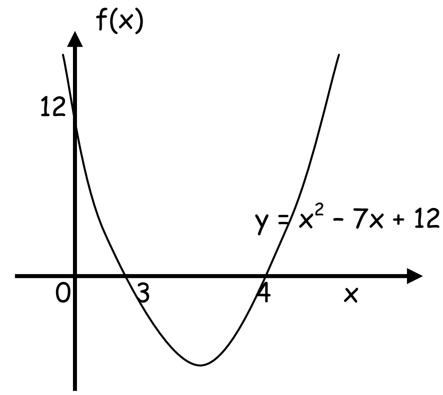
Thus the implied domain is $(-\infty, 5]$.

$h(x) = \sqrt{x-5} + \sqrt{8-x}$

h(x) is defined when $x - 5 \geq 0$, and $8 - x \geq 0$, i.e. when $x \geq 5$ when $x \leq 8$.
Thus the implied domain is [5, 8].

$f(x) = \sqrt{x^2 - 7x + 12}$

f(x) is defined when $x^2 - 7x + 12 \geq 0$.
$x^2 - 7x + 12 \geq 0$ is equivalent to $(x - 3)(x - 4) \geq 0$.
For this to be true, we need $(x - 3) \geq 0$ and $(x - 4) \geq 0$
$\therefore x \geq 4$ or $x \geq 3$.
For both of these to be true, x must be ≥ 4

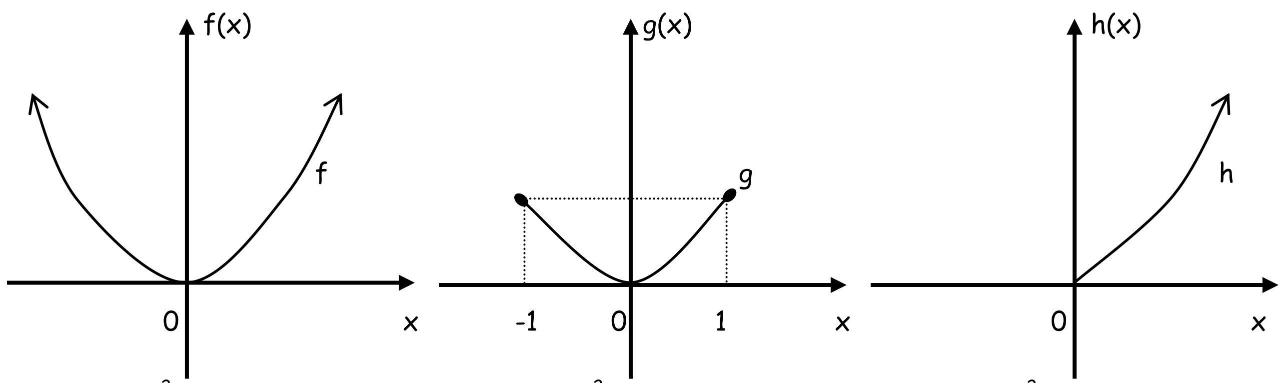


$(x - 3)(x - 4) \geq 0$, this is also true when
$(x - 3) \leq 0$ and $(x - 4) \leq 0$
$\therefore x \leq 4$ or $x \leq 3$.
For both of these to be true, x must be ≤ 4

Alternatively, we can just look at the graph, $f(x) \geq 0$ when $x \leq 3$ and when $x \geq 4$

Thus the implied domain is $[4, \infty) \cup (-\infty, 3]$

Example 7
Consider the functions



$f(x) = x^2, x \in R$ $g(x) = x^2, -1 \leq x \leq 1$ $h(x) = x^2, x \in R^+ \cup \{0\}$

The different letters, f, g, and h, used to name the functions, emphasise the fact that there are three different functions even though they have the same rule. They are different because they are defined for different domains. We call 'g' and 'h' restrictions of 'f' since their domains are subsets of the domain 'f'.

Useful signs

$\cup$	= union with	$\cap$	= intersection with	$\in$	= element of
$\subset$	= subset of (1st included in 2nd set)			$\notin$	= not an element of
$\varnothing$	= null set (empty set)				

Example 8

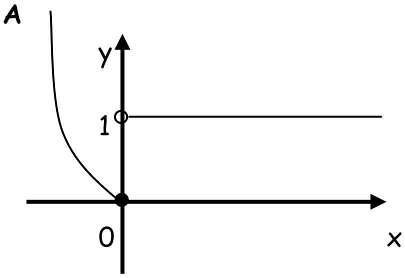
What is the domain of the function $f(x) = \frac{1}{\sqrt{x-2}}$?

The function exists when the denominator is > 0. $\therefore$ x needs to be > than 2.
This is written as $(2, \infty)$. Ensure that the brackets correct, they both need to be curved brackets.

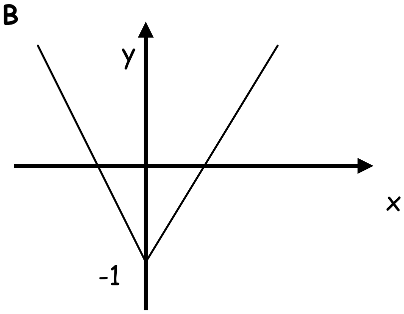
Example 9

Which one of the following relations is not a function?

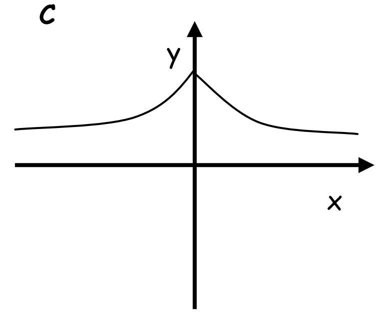
A



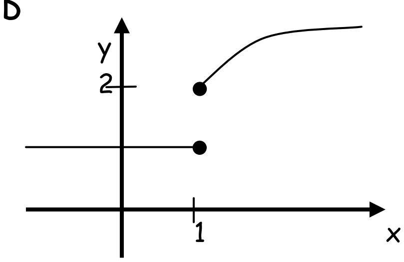
B



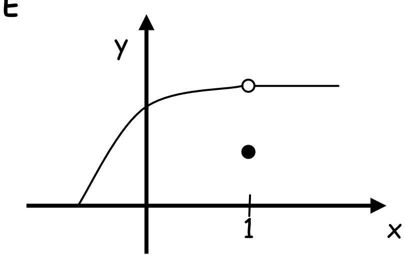
C



D



E



D, is not a function because the vertical line test fails at x = 1.

Example 10

What is the **range** of the function $g : R\backslash\{2\} \to R$, defined by $g(x) = \frac{1}{(x-2)^2} + 3$?

Note that the function is not defined when x = 2, since this would make g(x) include $\frac{1}{0}$.

$\frac{1}{(x-2)^2}$ is always positive, since it is squared, so as $x \to \pm\infty$, $\frac{1}{(x-2)^2} \to 0$

$$\therefore g(x) \to 0 + 3$$
$$\therefore g(x) \to 3$$

g(x) is always greater than 3, so the range can be defined as $\{y : y > 3\}$

Example 11

Which one of the following is not a one-to-one function?

A $f:[0, \infty) \to R$, where $f(x) = \sqrt{x}$
B $f:[0, \infty) \to R$, where $f(x) = x^2 - 1$
C $f:[0, \infty) \to R$, where $f(x) = 9$
D $f:[0, \infty) \to R$, where $f(x) = 3 - x^2$
E $f:[0, \infty) \to R$, where $f(x) = x(x^2 - 1)$

The best way to check your answer this question is to use your graphics calculator to plot the functions. Alternatively, and probably quicker, is to plot them in your head. Graph C is a many-to-one function. The others are all one-to-one functions.

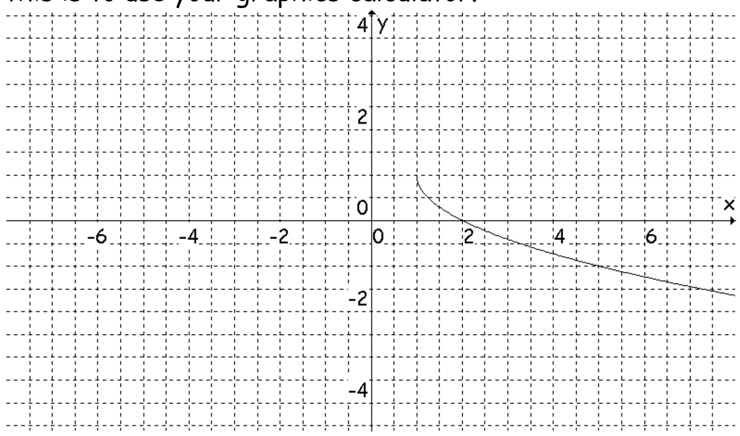
Example 12

What is the domain of the function $f(x) = \frac{2}{\sqrt{3-x}}$?

The domain exists when the square root is positive. $\therefore$ $x < 3$.
This domain is written as $(-\infty, 3)$. The brackets need to be curved, because $x \neq 3$.

Example 13

Draw the graph of the function $f(x) = 1 - \sqrt{x-1}$.
The quickest way to do this is to use your graphics calculator.

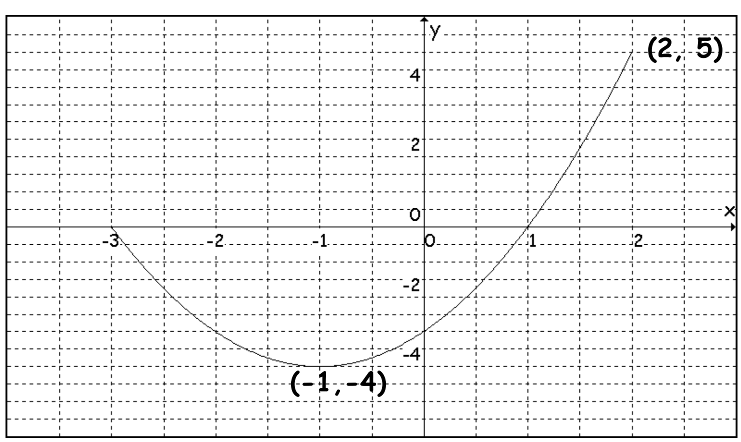


Example 14

For the function $f(-3, 2] \to R$, $f(x) = (x + 1)^2 - 4$ the range is

A (-3, 5]
B [-4, 5]
C (-4, 5]
D (0, 5]
E [-1, 2]

The graph looks like



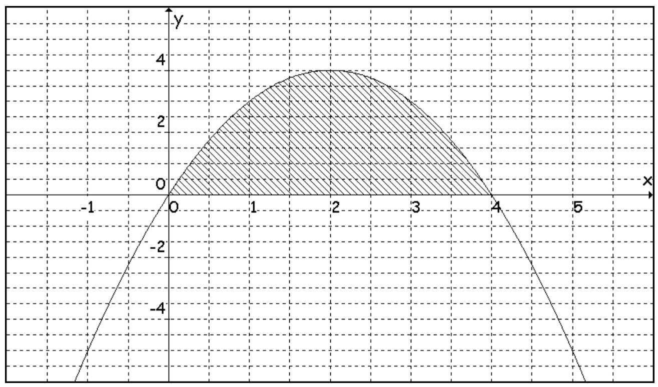
From Year 11 Methods, the turning point is (-1, 4), since this is a translation.
Using the domain as specified in the question, $f(-3) = (-2)^2 - 4 = 0$
$f(2) = 3^2 - 4 = 5.$

The minimum y value is –4 and the maximum y value is 5. $\therefore$ the range is [-4, 5] $\therefore$ B

Example 15

Determine the largest possible domain for the function $f(x) = \sqrt{4x - x^2}$

Draw the graph of $4x - x^2$



The function only exists when $4x - x^2 \geq 0$
$\therefore x(4 - x) \geq 0$
$\therefore 0 \leq x \leq 4$

The largest possible domain is [0, 4]